# Generalized center conditions and multiplicities for polynomial Abel equations of small degrees

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**Abstract.**

We consider an Abel equation $(*)$ $y' = p(x)y^2 + q(x)y^3$ with $p(x)$, $q(x)$ – polynomials in $x$. A center condition for this equation (closely related to the classical center condition for polynomial vector fields on the plane) is that $y_0 = y(0) \equiv y(1)$ for any solution $y(x)$. This condition is given by vanishing of all the Taylor coefficients $v_k(1)$ in the development $y(x) = y_0 + \sum_{k=2}^{\infty} v_k(x) y_0^k$. Following [BFY2] we introduce periods of the equation $(*)$ as those $\omega \in \mathbb{C}$, for which $y(0) \equiv y(\omega)$ for any solution $y(x)$ of $(*)$. The generalized center conditions are conditions on $p$, $q$ under which given $a_1, \ldots, a_k$ are (exactly all) the periods of $(*)$.

A new basis for the ideals $I_k = \{v_2, \ldots, v_k\}$ has been produced in [BFY1], defined by a linear recurrence relation. Using this basis and a special representation of polynomials, we extend results of [BFY2], proving for small degrees of $p$ and $q$ a composition conjecture, stated in [AL], [BFY2], [BFY3]. In particular, this provides transparent generalized center conditions in the cases considered. We also compute maximal possible multiplicity of the zero solution of $(*)$, extending results of [AL].

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## 1. Introduction

We consider the following formulation of the center problem (see e.g. [Sch] for a general discussion of the classical center problem): Let $P(x,y)$, $Q(x,y)$ be polynomials in $x$, $y$ of degree $d$. Consider the system of differential equations

$$\begin{cases} \dot{x} = -y + P(x,y) \\ \dot{y} = x + Q(x,y) \end{cases} \tag{1.1}$$

We say that a solution $x(t)$, $y(t)$ of (1) is closed if it is defined in the interval $[0,t_0]$ and $x(0)=x(t_0)$, $y(0)=y(t_0)$. We say that the system (1.1) has a center at 0 if all the solutions around zero are closed. Then the general problem is: under what conditions on $P,Q$ the system (1.1) has a center at zero?

It was shown in [Ch] that one can reduce the system (1.1) with homogeneous $P$, $Q$ of degree $d$ to the Abel equation

$$y' = p(x)y^2 + q(x)y^3 \tag{1.2}$$

where $p(x)$, $q(x)$ are polynomials in $\sin x$, $\cos x$ of degrees depending only on $d$. Then (1.1) has a center if and only if (1.2) has all the solutions periodic on $[0,2\pi]$, i.e. solutions $y=y(x)$ satisfying $y(0)=y(2\pi)$.

We will look for solutions of (1.2) in the form

$$y(x,y_0) = y_0 + \sum_{k=2}^{\infty} v_k(x,\lambda)y_0^k, \tag{1.3}$$

where $y(0,y_0)=y_0$. The coefficients $v_k$ turn out to be polynomials both in $x$ and $\lambda$, where $\lambda=(\lambda_1,\lambda_2,\ldots)$ is the (finite) set of the coefficients of $p$, $q$. Shortly we will write $v_k(x)$.

Then $y(2\pi)=y(2\pi,y_0)=y_0+\sum_{k=2}^{\infty} v_k(2\pi)y_0^k$ and hence the condition $y(2\pi)\equiv y(0)$ is equivalent to $v_k(2\pi)=0$ for $k=2,3,\ldots\infty$,

Consider an ideal $J=\{v_2(2\pi),v_3(2\pi),\ldots v_k(2\pi),\ldots\}\subseteq \mathbb{C}[\lambda]$. By Hilbert Basis theorem there exists $d_0<\infty$, s.t. $J=\{v_2(2\pi),v_3(2\pi),\ldots v_{d_0}(2\pi)\}$. After determination of $d_0$ the general problem will be solved, since we get a finite number of conditions on $\lambda$, which define the set of $p$, $q$ having all the solutions closed . The problem is that the Hilbert theorem does not allow us to define $d_0$ constructively.

As it was shown in [AL], [L], [BFY1-3] there are good reasons to consider the equation (1.2) with $p$, $q$ usual polynomials instead of trigonometric ones (although the relation to the initial problem (1.1) becomes less direct here). In this paper we restrict ourselves to this case, although some of our results remain valid for the trigonometric case. Notice, however, that the composition conjecture, as it is stated below, is not true in the trigonometric case (see [A]).

## 2. Composition conjecture, objectives and results.

In what follows we shall study the Abel equation (1.2) with $p$, $q$ the usual polynomials in $x$ instead of trigonometric ones. In this case we say that the equation (1.2) defines a center if $y(1, y_0) \equiv y_0$. Although this property does not correspond to the initial problem (1.1), it presents an interest by itself and it has been studied in [GL], [L], [AL] and in many others papers.

Let us study instead of $J \subseteq \mathbb{C}[\lambda]$ the polynomials ideal $I \subseteq \mathbb{C}[\lambda, x]$,

$$I = \{v_2(x), v_3(x), \ldots v_k(x), \ldots\} = \bigcup_{k=2}^{\infty} I_k, \text{ where } I_k = \{v_2(x), v_3(x), \ldots v_k(x)\}.$$

The classical problem is to find conditions on $p$, $q$, under which $x = 1$ is a common zero of all $I_k$.

Our **generalized center problem** is the following:
*For a given set of different complex numbers $a_1 = 0, a_2, \ldots, a_\ell$ find conditions on $p$, $q$, under which these numbers are common zeroes of $I$.*

We shall say that such $p$, $q$ **define a center on** $[0; a_2; \ldots; a_\ell]$, or that they satisfy **generalized center conditions**. Numbers $a_2, \ldots, a_\ell$ will be called **periods** of (1.2), since $y(0) = y(a_i)$ for all the solutions $y(x)$ of (1.2).

In contrast to the situation over the real segment [0,1], the condition $y(0) = y(\omega)$ over the complex plane requires an additional explanation. Indeed, the solutions of (1.2) have "moving singularities', where the solution behaves roughly as $(x - x_0)^{-1/2}$. Hence the value $y(\omega)$ depends on the path along which we continue it from $y(0)$.

However, for $y_0$ sufficiently small, the singularities of $y(x)$, satisfying $y(0) = y_0$, can be shown to be out of any prescribed disk around the origin in the $x$-plane. (Notice that $y \equiv 0$ is a solution of (1.2).) Hence the values $y(\omega)$ for $y_0$ small can be defined independently of the chosen continuation path. Our precise center property is that the germ $y(\omega)(y_0)$ at $y_0 = 0$ is identically equal to $y_0$. (Of course, if this happens, by analytic continuation $y(\omega)$, properly defined, is always equal to $y_0$.)

For each number $\omega$ we define **the multiplicity of the zero solution with respect to** $\omega$. Here we follow notations in [AL], where multiplicity was defined for the standard equation (1.2) on $[0, \omega] \subset \mathbb{R}$ as the number $\mu$ such that $v_1(\omega) = 1$, $v_2(\omega) = v_3(\omega) = \ldots = v_{\mu-1}(\omega) = 0$, and $v_\mu(\omega) \neq 0$. Now we extend this notation and define multiplicity for the equation (1.2) on $\mathbb{C}$ for any number $\omega \in \mathbb{C}$.

The number $\omega$ will be a period if $v_1(\omega) = 1$ and $v_k(\omega) = 0$ for all $k > 0$. In other words, $\omega$ is a period, if its multiplicity is equal to infinity.

We define multiplicity $\mu(d_1, d_2)$ as the maximal value of multiplicity achieved for some $p$, $q$ of degrees $d_1 - 1$, $d_2 - 1$ respectively and for some $\omega \in \mathbb{C}$.

Notice that here and below we define the degree of a polynomial as the highest degree of $x$ in it, entering with **nonzero** coefficient.

The following **composition conjecture** has been proposed in [BFY2]:

$I = \bigcup_{k=1}^{\infty} I_k$ **has zeroes** $a_1, a_2, \ldots a_k$ **,** $a_1 = 0$**, if and only if**

$P(x) = \int_0^x p(t)dt = \tilde{P}(W(x))$ , $Q(x) = \int_0^x q(t)dt = \tilde{Q}(W(x))$ ,

**where** $W(x) = \prod_{i=1}^{k}(x - a_i)\tilde{W}$ **is a polynomial, vanishing at** $a_1, a_2, \ldots a_k$**, and** $\tilde{P}$**,** $\tilde{Q}$ **are some polynomials without free terms.**

Sufficiency of this conjecture is almost obvious (see [BFY2]). But we still do not have any method to prove the necessity of this conjecture in the general case, although the connection between this conjecture and some interesting analytic problems was established (see [BFY1], [BFY2], [BFY3]), and for some simplified cases it was partially or completely proved.

Notice that if the composition conjecture would be true it could provide compact and transparent generalized center conditions (which relatively easily can be expressed by explicit equations on the coefficients of $p$ and $q$). See [BFY2] and section 7 below for explicit formulae.

As for now the only known to us way to prove the conjecture is to compute consequently polynomials $v_n(x)$'s, to solve systems of polynomial equations $v_n(a_j) = 0$ in many variables ($a_j$ and coefficients of $p$, $q$), and to show that the solutions satisfy composition conjecture.

In [BFY2] it was shown that the composition conjecture is true for the cases $(\deg P, \deg Q) = (d_1, d_2) = (2,2) - (2,6)$ and $(3,2)$, $(3,3)$. In [AL] multiplicities were computed for the cases $(\deg P, \deg Q) = (d_1, d_2) = (2,2) - (2,6)$ and (3,4).

In this paper we present the following results:

a) The maximal number of different zeroes of $I$, i.e. the maximal number of periods of (1.2) is estimated (section 3).

b) The generalized center conditions are obtained for some classes of polynomials $p$, $q$ (of a special form but of an arbitrarily high degree) (sections 4, 5)

c) The composition conjecture is verified for the following additional cases $(\deg P, \deg Q) = (d_1, d_2) = (2,7), (3,4), (4,2), (4,3), (4,4), (5,2), (6,2), (3,6)$. It is done using computer symbolic calculations with some convenient representation of $P$ and $Q$ . For these and previous cases multiplicities are computed (section 6).

d) On this base explicit center conditions for the equation (1.2) on [0,1] are written in all the cases considered. They turn out to be very simple and transparent, especially in comparison with the equations provided by vanishing of $v_k(1, \lambda)$ (section 7).

**3. Maximal number of different zeroes of $I$.**

One can easily show (by substitution of the expansion (1.3) into the equation (1.2)) that $v_k(x)$ satisfy recurrence relations

$$\begin{cases} v_0(x) \equiv 0 \\ v_1(x) \equiv 1 \\ v_n(0) = 0 \quad \text{and} \\ v_n'(x) = p(x) \sum\limits_{i+j=n} v_i(x)v_j(x) + q(x) \sum\limits_{i+j+k=n} v_i(x)v_j(x)v_k(x), \ n \geq 2 \end{cases} \tag{3.1}$$

It was shown in [BFY1] that in fact the recurrence relations (3.1) can be linearized, i.e. the same ideals $I_k$'s are generated by $\{\psi_1, \dots \psi_k\}$, where $\psi_k(x)$ satisfy linear recurrence relations

$$\begin{cases} \psi_0(x) \equiv 0 \\ \psi_1(x) \equiv 1 \\ \psi_n(0) = 0 \quad \text{and} \\ \psi_n'(x) = -(n-1)\psi_{n-1}(x)p(x) - (n-2)\psi_{n-2}(x)q(x), \ n \geq 2 \end{cases} \tag{3.2}$$

which are much more convenient then (3.1). We call (3.2) **the main recurrence relation.**

Direct computations (including several integrations by part) give the following expressions for the first polynomials $\psi_k(x)$, solving the recurrence relation (3.2) (remind that $P(x) = \int_0^x p(t)dt$, $Q(x) = \int_0^x q(t)dt)$ :

$$\psi_2(x) = \ - P(x)$$
$$\psi_3(x) = P^2(x) - Q(x)$$
$$\psi_4(x) = \ - P^3(x) + 3P(x)Q(x) - \int_0^x q(t)P(t)dt$$

Consequently, we get the following set of generators for the ideals $I_k$, $k = 2, \dots, 4$

$$I_2 = \{P\}, \ I_3 = \{P, Q\}, \ I_4 = \{P, Q, \int qP\}.$$

Therefore, if $a$ is a zero of the ideal $I_4$, it must satisfy the following equations:

$$P(a) = 0 \ , \ Q(a) = 0, \ \int_0^a P(t)q(t)dt = 0$$

Let us assume now that the set of zeroes of $I_4$ consists of the points $a_1 = 0, a_2, \dots, a_\nu$, $a_i \neq a_j$. In particular, $a_i$ are common zeroes of $P$ and $Q$, and we can write

$$P(x) = W(x)P_1(x), \ Q(x) = W(x)Q_1(x)$$

where $W(x) = \prod_{i=1}^{\nu}(x - a_i)$.

Substituting these representations into the equation $\int_0^a P(t)q(t)dt = 0$ and integrating by parts, we get for $i = 1, \ldots, \nu$,

$$\int_0^{a_i} W^2(p_1Q_1 - P_1q_1) = 0$$

Here $p_1(x) = P_1'(x)$, $q_1(x) = Q_1'(x)$.

This allows us to prove the following theorem:

**Theorem 3.1.** *Either the number of different zeroes (including* 0*) of $I$ is less than or equal to* $(\deg P + \deg Q)/3$*, or $P$ is proportional to $Q$.*

**Proof:**

Let $P = WP_1$, $Q = WQ_1$, $W = \prod_{i=1}^{k}(x - a_i)$ – a polynomial, accumulating all surviving zeroes $a_1, \ldots, a_k$, $\deg P_1 = \ell_1$, $\deg Q_1 = \ell_2$. Consider the function $f(x) = \int_0^x W^2(p_1Q_1 - q_1P_1)dt$ and assume first that $p_1Q_1 - q_1P_1 \neq 0$. Since all $a_j$ are zeroes of both $f$ and $W$, we get $f(x) = W^3S(x)$, hence $\deg f(x) \geq 3k$. From the other side $\deg f(x) = 2k + (\ell_1 + \ell_2 - 1) + 1 = \ell_1 + \ell_2 + 2k$. So, $\ell_1 + \ell_2 + 2k = (\ell_1 + k) + (\ell_2 + k) = \deg P + \deg Q \geq 3k$, q.e.d.

Now let $p_1Q_1 - q_1P_1 = 0$, i.e. $(P_1Q_1)' = 2q_1P_1$. Denote $P_1Q_1$ by $X$, $Q_1$ by $Y$. Then $q_1 = Y'$, $P_1 = X/Y$, hence $X' = 2Y'\frac{X}{Y}$, i.e. $\frac{X'}{X} = 2\frac{Y'}{Y}$, i.e. $X = CY^2$, i.e. $P_1Q_1 = CQ_1^2$, q.e.d.

**Corollary 3.2.** *Either $P$ is proportional to $Q$, or the number of different periods of (1.2) is less than or equal to* $((\deg P + \deg Q)/3) - 1$*.*

**Remark:** This result is implicitly contained in computations, given in [BFY3] .

## 4. A convenient representation of $P$ and $Q$ and algebra of compositions of polynomials.

Let polynomials $r(x)$ and $W(x)$ be given. Assume we are interested in checking whether $R(x) = \int_0^x r(t)dt$ can be represented as a composition with $W(x)$, i.e. if $R(x) = \tilde{R}(W(x))$ for some polynomial $\tilde{R}$ without free term.

Let $W(x) = x(x - a)$. Notice, that the derivative of W is a polynomial of the first degree $W'(x)$, the polynomial $W(x)W'(x)$ has the third degree and so on. Generally, polynomials $W(x)^k$ have degree $2k$ and polynomials $W(x)^k\ W'(x)$ have degree $2k + 1$.

Therefore they are linearly independent and form a basis of $\mathbb{C}[x]$. So, one can uniquely represent any polynomial $r(x)$ as a linear combination of polynomials $W(x)^k$ and $W(x)^k W'(x)$. Hence any polynomial $r(x)$ of the degree $2k$ or $2k+1$ we will write in the form

$$r(x) = W(x)^k\ (\alpha_k W(x)' + \beta_k)\ +\ W(x)^{k-1}\ (\alpha_{k-1} W(x)' + \beta_{k-1})\ +\ \dots\ +\ (\alpha_0 W(x)' + \beta_0),$$

or simply

$$r(x) = W^k(\alpha_k W' + \beta_k) + W^{k-1}(\alpha_{k-1} W' + \beta_{k-1}) + \dots + (\alpha_0 W' + \beta_0).$$

Generally, if $W(x) = x(x-a_2)\dots(x-a_\ell)$, $\deg W(x) = \ell$ and $r(x)$ is a polynomial of degree $m\ell + k$, $k \in \{0, \dots, \ell-1\}$, then $r(x)$ can be uniquely represented in the form

$$r = W^m(c_m^1 W' + c_m^2 W'' + \dots + c_m^k W^{(\ell)}) + \dots + (c_0^1 W' + c_0^2 W'' + \dots + c_0^k W^{(\ell)}),$$

(where, of course, $W^{(\ell)}$ is a constant).

Now we can state the following

**Theorem 4.1** $R(x) = \int_0^x r(t)dt$ *is a composition with* $W(x)$ *if and only if* $c_j^i = 0$ *for* $i \geq 2$, $j = 0, \dots, m$.

**Proof:**

If $c_j^i = 0$ for $i \geq 2$, $j = 0, \dots, m$, then obviously $R(x)$ is a composition with $W(x)$.

Let $R(x)$ be a composition with $W(x)$, then $r(x) = R'(x) = \tilde{R}(W)W'$, and by uniqueness of basis expansion all $c_j^i = 0$ for $i \geq 2$, q.e.d.

Notice that in the case $\deg W = \ell > 2$ we can instead of the basis $\{W^n W^{(k)}, k = 0, \dots, \ell-1\}$ consider the basis $\{W^n W', W^n x^k, k = 0, \dots, \ell-2\}$ and the same statement holds.

This representation will be used below for the verification of the composition conjecture (see section 6).

## 5. Generalized center conditions for some classes of polynomials.

The representation, introduced in the section 4, gives us a convenient tool for finding generalized center conditions for some classes of polynomials, i.e. for the verification of the composition conjecture. As the first example let us show that one can easily produce sequences of polynomials $p$ and $q$ of arbitrarily high degrees, for which the composition conjecture is true, i.e the generalized center conditions imply the representability of $P$, $Q$ as a composition.

Let $a_1 = 0, a_2, \dots, a_\ell$ be given. Consider any polynomial $W(x)$ vanishing at all the points $a_j$, $\quad j = 1, \dots, \ell$.

**Theorem 5.1** *Assume that for at least one $a_j$, $\int_0^{a_j} W^k dx \neq 0$ and $\int_0^{a_j} W^n dx \neq 0$. Polynomials $p = W^k(\alpha + \beta W')$, $q = W^n(\gamma + \delta W')$ define center on $[0; a_1; \ldots; a_\ell]$ if and only if $\alpha = \gamma = 0$.*

**Remark:** Notice, that the condition "$\int_0^{a_j} W^k dx \neq 0$ for at least one $a_j$" is satisfied, for instance, for $W(x) = \prod_{i=1}^{\ell}(x - a_i)$, where all $a_j$ are different. Indeed, consider the function $f(x) = \int_0^x W(t)^k dt$. If all $a_j, j = 1, \ldots, \ell$ would be zeroes of $f(x)$, then $\deg f \geq (k+1)\ell$. But $\deg W = \ell$, so $\deg f(x) = k\ell + 1$. We obtain $k\ell + 1 \geq (k+1)\ell$, which is not satisfied for $\ell > 1$ .

Similarly one can show that $W(x) = \prod_{i=1}^{\ell}(x - a_i)^{m_i}$ satisfies the condition "$\int_0^{a_j} W^k dx \neq 0$ for at least one $a_j$" for almost all $k$, and so on. So, this condition is "almost generic".

**Proof of theorem 5.1:** Since $\psi_2(x) = P(x)$, the conditions $\psi_2(a_j) = 0$ imply $\alpha = 0$. Since $\psi_3(x) = P^2(x) - Q(x)$, the conditions $\psi_3(a_j) = 0$ imply $\gamma = 0$, q.e.d.

**Theorem 5.2** *Assume that $\deg W > 2$ and for at least one $a_j$*

$$\det \begin{vmatrix} \int_0^{a_j} W^n dx & \int_0^{a_j} W^n W'' dx \\ \int_0^{a_j} W^{n+k+1} dx & \int_0^{a_j} W^{n+k+1} W'' dx \end{vmatrix} \neq 0.$$

*Polynomials $p = W^k(\alpha + \beta W')$, $q = W^n(\gamma + \delta W' + \epsilon W'')$ define center on $[0; a_1; \ldots; a_\ell]$ if and only if $\alpha = \gamma = \epsilon = 0$.*

**Proof:** The conditions $\psi_2(a_j) = 0$ imply $\alpha = 0$. The conditions $\psi_3(a_j) = 0$ imply

$$\gamma \int_0^{a_j} W^n + \epsilon \int_0^{a_j} W^n W'' = 0,$$

and the conditions $\psi_4(a_j) = 0$ imply

$$\gamma \int_0^{a_j} W^{n+k+1} + \epsilon \int_0^{a_j} W^{n+k+1} W'' = 0.$$

If the determinant of the system is nonzero, we get that the system has the only zero solution, q.e.d.

**Remark:** In this article we discuss questions, connected to the polynomial case, but actually constructions from section 5 can be easily generalized to the case of arbitrary (analytic) functions $p$, $q$, $W$

## 6. Verification of the main conjecture and counting of multiplicities.

### 6.1. Remarks about rescaling of P and Q.

**1)** As it was stated above, we always assume that the highest degree coefficient is not zero.

**2)** As it was shown in [BFY2], if $\deg Q \neq 2\deg P$ then using rescaling $x \mapsto C_1 x$, $y \mapsto C_2 y$, one can make the leading coefficients of $P$, $Q$ be equal to any positive number. So, for possible cases we will use polynomials $P$ and $Q$ in the form where the leading coefficients equal either 1 or 2. For instance, for the case deg P=3, deg Q=4 we will assume that $P(x) = 2x^3 + \ldots$ (terms of degree less than 3), $Q(x) = x^4 + \ldots$(terms of degree less than 4) and so on.

### 6.2. Main results.

**Theorem 6.1** *The following table of the maximal possible values of multiplicity $\mu(d_1, d_2)$ holds:*

| $d_1 = \deg P$ / $d_2 = \deg Q$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 2 | 3 or $\infty$ | 4 | 4 or $\infty$ | 8 | 9 or $\infty$ |
| 3 | 4 | 4 or $\infty$ | 8 | | |
| 4 | 4 or $\infty$ | 8 | 9 or $\infty$ | | |
| 5 | 5 | | | | |
| 6 | 5 or $\infty$ | 10 or $\infty$ | | | |
| 7 | 10 | | | | |

In this theorem we extend results of [AL], where multiplicities for the equation (1.2) on [0,1] were computed for the cases $(\deg P, \deg Q) = (2,2) - (2,6), (3,4)$. Alwash and Lloyd used the standard representation of polynomials in basis $\{x^n, n = 0, 1, \ldots\}$ on [0,1] and leading coefficients of $P$ and $Q$ as parameters. Also they used nonlinear recurrence relation (2.1). Our representation together with linear recurrence relation (2.2) allows us to go further and to compute multiplicities for higher degrees of $P$ and $Q$.

**Theorem 6.2** *For these cases the composition conjecture is true and the following table gives the possible number of different periods in each case:*

| deg Q \ deg P | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 2 | 0, 1 | 0 | 0, 1 | 0 | 0, 1 |
| 3 | 0 | 0, 2 | 0 | | |
| 4 | 0, 1 | 0 | 0, 1, 3 | | |
| 5 | 0 | | | | |
| 6 | 0, 1 | 0, 2 | | | |
| 7 | 0 | | | | |

**Proof of theorems 6.1 and 6.2:** The proof consists of computations of $\psi_n(x)$ for each of the cases considered, and solving the systems of polynomial equations. It was done using computer symbolic calculations using the special representation of $P$ and $Q$. Descriptions of computations for the most interesting cases are given below:
deg P=4, deg Q=4 – subsection 6.4
deg P=3, deg Q=6 – subsection 6.5

Other cases were considered similarly, but in most of the cases straightforward computations were far beyond the limitations of the computer used. Consequently, some non-obvious analytic simplifications were used. Part of them is presented in 6.4 - 6.5

Computations for the cases $(\deg P, \deg Q) = (2,7), (3,6)$ were performed together with Jonatan Gutman and Carla Scapinello.

### 6.3 Remark about resultants.

Resultants give us a convenient tool for checking, whether $n+1$ polynomials of $n$ variables $P_i(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ do not have common zeroes.

Consider one example. Assume we are interested whether polynomials $P(x,y)$, $Q(x,y)$, $R(x,y)$ have common zeroes.

**Claim.** Let $\texttt{Resultant}[P,Q,x] = S_1(y)$, $\texttt{Resultant}[R,Q,x] = S_2(y)$. If $\texttt{Resultant}[S_1,S_2,y] \neq 0$, then $P$, $Q$, $R$ do not have common zeroes.

**Proof:** Assume there exists common zero $(x_0,y_0)$ of all polynomials $P,Q,R$, then $S_1(y_0) = S_2(y_0) = 0$, hence $\texttt{Resultant}[S_1,S_2,y] = 0$. Contradiction.

The general construction for $n+1$ polynomials of $n$ variables is exactly the same.

### 6.4 deg P=4, deg Q=4

Our goal is to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has common zeroes others than 0 if and only if either $P(x) = \tilde{P}(W(x))$, $Q(x) = \tilde{Q}(W(x))$ for certain polynomials $\tilde{P}$, $\tilde{Q}$ without free terms, where $W(x) = x(x-a)$, $a \neq 0$, or $P$ is proportional to $Q$ (and in this case $W = P$ and again $P = \tilde{P}(W)$, $Q = \tilde{Q}(W)$). In the process of computations we find the maximal finite multiplicity, which is achieved on polynomials unrepresentable as a composition.

**1)** If $P$, $Q$ are proportional, we are done. If $P$, $Q$ are not proportional, then from the theorem 3.1. we get that the maximal number of different zeroes is 2. And one of them is necessarily 0.

**2)** Assume that $I$ has zeroes 0, $a$ ($a \neq 0$). Since zeroes of $I$ should be also zeroes of $P$ and $Q$ , $P$ and $Q$ can be represented in the form (up to rescaling)

$$P = W(W + \gamma W' - \alpha), \quad Q = W(W + \delta W' - \beta)$$

where $W = x(x-a)$. For such $P$, $Q$ numbers 0, $a$ are common zeroes of ideals $I_1$, $I_2$, $I_3$. Then we will directly calculate, using the "Mathematica" software, ideals $I_4$– $I_8$ and we will show that the only possibilities for $I$ to have zeroes 0, $a$ are either $\gamma = \delta = 0$ or $P = Q$. It will complete the verification of the composition conjecture for this case.

**3)** Running a program, which utilizes recurrence relation (3.2)

```
(*   n-the number of ideals to be computed   *)
(*   P, Q are defined symbolically           *)
     W=x(x-a);
     P=W*(W + gamma W' - alpha);
     Q=W*(W + beta W' - delta);
     psi[0]=0;
     psi[1]=1;
     psi[2]=-P;
     Do[psi[i]=Integrate[
         -(i-1)psi[i-1]*p-(i-2)psi[i-2]*q,x],{i,3,n}];
     x=a;
     Do[Simplify[psi[i]],{i,1,n}];
```

we obtain the following results:

$$\psi_4(a) = \frac{a^5\ (7\,\alpha\,\delta + 2\,a^2\ (\delta - \gamma) - 7\,\beta\,\gamma)}{210},$$

$$\psi_5(a) = \frac{a^7\ (4\,a^4\ (\delta - \gamma) + 66\,\alpha\ (\alpha\,\delta - \beta\,\gamma) + 11\,a^2\ (3\,\alpha\,\delta - 2\,\alpha\,\gamma - \beta\,\gamma))}{6930}.$$

Since $a \neq 0$, we get

$$\frac{2}{7}a^2(\delta - \gamma) = \beta\gamma - \alpha\delta \qquad (*)$$

$$(4a^4 + 22a^2\alpha)(\delta - \gamma) + (\alpha\delta - \beta\gamma)(66\alpha + 11a^2) = 0 \qquad (**)$$

**a)** If $\delta = \gamma \neq 0$, then from $(*)$ $\alpha = \beta$, and hence $P = Q$, we are done.

**b)** If $\delta = \gamma = 0$, then we get a composition with $W$, and we are done.

**c)** Assume now $\delta \neq \gamma$. Let us prove that in this case polynomials $\psi_k(a)$, $k = 6, 7, 8, 9$ can not have common zeroes. Substituting $\alpha\delta - \beta\gamma = \frac{2a^2}{7}(\gamma - \delta)$ into $(**)$ and dividing it by $\gamma - \delta$ we obtain $\alpha = -\frac{3a^2}{11}$. Then from $(*)$ we get $\delta = \frac{77\beta\gamma}{a^2} + 22\gamma$. Running the program for these values of $\alpha$, $\delta$, we get

$$\psi_6(a) = \frac{-\left(a^7\ (3\,a^2 + 11\,\beta)\ \gamma\ (-4719\,a^2 + 3\,a^6 - 17303\,\beta - 363\,a^4\,\gamma^2)\right)}{10900890}.$$

If $\gamma = 0$, then from $(*)$ we get $\frac{2a^2\delta}{7} + \alpha\delta = 0$ , i.e. $\delta(\frac{2a^2}{7} + \alpha) = 0$. Since $\alpha = -\frac{3a^2}{11}$, we get $\delta = 0$. Contradiction to the assumption $\delta \neq \gamma$.

If $\beta = -\frac{3a^2}{11}$, then $\beta = \alpha$, hence from $(*)$ $\delta = \gamma$. Contradiction.

Otherwise from $\psi_6(a) = 0$ we get

$$\beta = \frac{3a^6 - 4719a^2 - 363a^4\gamma^2}{17303}, \qquad (***)$$

and running the program for these values (i.e. after substitution of $\alpha$, $\delta$, $\beta$), we get that $\psi_7(a)$, $\psi_8(a)$, $\psi_9(a)$ are polynomials in $a$ and $\gamma$ times $(\gamma(a - 11\gamma)(a + 11\gamma)$. If $\gamma = \pm a/11$, then from $(***)$ we get $\beta = -3a^2/11$, so $\alpha = \beta$ and hence $\gamma = \delta$. Contradiction. Notice that $\gamma \neq 0$, since in this case $\delta = 0$ - contradiction.

So, we get 3 polynomials of two variables $\gamma$, $a$ - reminders after division polynomials $\psi_7(a), \psi_8(a), \psi_9(a)$ by $(a^2 - 121\gamma^2)$. Canceling constants and computing resultants, we get nonzero number, q.e.d.

The maximal finite multiplicity is 9 and it is achieved on the polynomials

$$P = x(x-a)\left(x^2 + (2\gamma - a)x + \frac{3a^2}{11} - a\gamma\right)$$

$$Q = x(x-a)(W + \delta W' - \beta)$$

where

$$\beta = \frac{3a^6 - 4719a^2 - 363a^4\gamma^2}{17303},$$

$$\delta = \gamma + \frac{21a^4\gamma}{1573} - \frac{21a^2\gamma^3}{13},$$

and $a, \gamma$ are chosen to vanish $\psi_7(a), \psi_8(a)$.

## 6.5 deg P=3, deg Q=6

This subsection is one of the most interesting parts of our computations, since for the first time from the theorem 3.1. follows, that the number of different zeroes may be either 2 or 3. The nontrivial common divisor of 3 and 6 is equal to 3, and we have to prove that in this case $I = \bigcup_{k=1}^{\infty} I_k$ has common zeroes others than 0 if and only if $Q(x)$ can be represented as a composition with $W(x) = Const\, P(x)$. The next point why this case differs from others is that according to the subsection 6.1 we can assume that only one of leading coefficients of $P$, $Q$ is 1. Say, the leading coefficient of $Q$ is 1, and the leading coefficient of $P$ is $\lambda$.

**1)** Assume first, that there are two common zeroes 0, $a$, which means that we can put

$$P = \lambda W(x+\alpha), \qquad Q = W(W^2 + \beta x^3 + \gamma x^2 + \delta x + \epsilon),$$

where $W(x) = x(x-a)$.

**a)** Let $a + 2\alpha \neq 0$. After running a "Mathematica" program up to $\psi_5(a)$, we express

$$\epsilon = \frac{-a^4 + 5a^3\beta + 12a^2\alpha\beta + 14\alpha\delta + 4a^2\gamma + 14a\alpha\gamma}{14}$$

$$\gamma = -\frac{-9a^3 - 18a^2\alpha + 31a^2\beta + 44a\alpha\beta - 22\alpha^2\beta}{22(a+2\alpha)}$$

After substitution and running the program again, we obtain from $\psi_6(a) = 0$ that $\beta = a + 2\alpha$. After substitution of all these values into expression for $Q$, we obtain

$$Q = W(W + (a+2\alpha)x^3 + (-a^2 - 2a\alpha + \alpha^2)x^2 + \delta x + \alpha(a\alpha^2 + \delta))$$

$$= \left(W(x+\alpha)\right)\left(W(x+\alpha) + a\alpha^2\delta)\right),$$

which means that we get the composition, and $\alpha$ is necessary the zero of our ideal.

We would like to stress, that we have obtained that $\alpha$ is a zero of $I$ without direct checking conditions $\psi_k(\alpha) = 0$.

**b)** For $\alpha = -a/2$ we obtain from $\psi_4(a) = 0$ that

$$\epsilon = -\frac{a^4 + a^3\beta + 7a\delta + 3a^2\gamma}{14},$$

after that we immediately get from $\psi_5(a) = 0$ that $\beta = 0$. Then

$$\psi_6(a) = Const_1\, a^{11}\lambda(-a^2 + 4\gamma)(-20a^2 + 52\gamma - 21a^2\lambda^2)$$

Let $\gamma = a^2/4$. After substitution of $\alpha, \beta, \gamma, \epsilon$ into the expression for $Q$, we get $Q = (W(x - a/2))(W(x - a/2) + a^3 + 4\delta)$, so the composition conjecture holds with $x(x-a)(x-a/2)$ as the greater common divisor of $P$ and $Q$ in the composition algebra of polynomials.

Now let $\gamma \neq a^2/4$. Then from $\psi_6(a) = 0$ we obtain $\gamma = \dfrac{20a^2 + 21a^2\lambda^2}{52}$. After substituting and performing computations, we get

$$\psi_7(a) = \frac{-a^{15}\,\lambda^2\,(1+3\,\lambda^2)\,(52\,\delta + a^3\,(20+21\,l^2))}{487206720}.$$

If $\lambda = \pm\dfrac{i}{\sqrt{3}}$, then $\gamma = \dfrac{a^2}{4}$. Contradiction. So, we express $\delta = \dfrac{-a^3(20+21\lambda^2)}{52}$. Substituting it into the program and computing $\psi_8(a)$, we get

$$\psi_8(a) = \frac{a^{21}\,\lambda\,(9+29\,\lambda^2+5508\,\lambda^4+16506\,\lambda^6)}{12274686103680}.$$

The equation $\psi_8(a) = 0$ has the solutions $\lambda = 0$, $\lambda^2 = -1/3$, $\lambda^2 = \dfrac{-1 \pm 13i\sqrt{293}}{5502}$.

For $\lambda = \pm\dfrac{i}{\sqrt{3}}$ we have $\gamma = \dfrac{a^2}{4}$. Contradiction. And for $\lambda^2 = \dfrac{-1 \pm 13i\sqrt{293}}{5502}$ we obtain that $\psi_8(a) = 0$, $\psi_9(a) = 0$, but $\psi_{10}(a) \neq 0$, q.e.d.

The maximal finite multiplicity is 10 and it is achieved on polynomials

$$P = \lambda x(x-a)\left(x - \frac{a}{2}\right)$$

$$Q = x(x-a)\left(x^4 - 2ax^3 + \frac{a^2(72+21\lambda^2)}{52}x^2 - \frac{a^3(20+21\lambda^2)}{52}x + \frac{a^4(1+3\lambda^2)}{26}\right),$$

where $\lambda^2 = \dfrac{-1 \pm 13i\sqrt{293}}{5502}$.

**2)** Now comes another interesting case, when we assume from the very beginning that we have three distinct common zeroes $0, a, b$. Here we put

$$P = \lambda W, \quad Q = W(W + \alpha x^2 + \beta x + \gamma), \text{ where } \quad W = x(x-a)(x-b).$$

Notice, that here in contrast to the all previous computations we must check vanishing at the two different points $a, b$. The equations $\psi_4(a) = 0$, $\psi_4(b) = 0$ form linear system with respect to $\alpha, \beta$:

$$\begin{cases} \dfrac{-a^5\,\lambda\,(5\,a^3\,\alpha + 14\,a\,b\,(\alpha\,b - \beta) + 14\,b^2\,\beta + 4\,a^2\,(-4\,\alpha\,b + \beta))}{840} = 0 \\ \dfrac{-b^5\,\lambda\,(14\,a^2\,(\alpha\,b + \beta) + b^2\,(5\,\alpha\,b + 4\,\beta) - 2\,a\,b\,(8\,\alpha\,b + 7\,\beta))}{840} = 0 \end{cases}.$$

The determinant of this system is equal to $70(a-b)^5$, so for $a \neq b$, $\lambda \neq 0$, $a \neq 0$, $b \neq 0$ the system may have the only zero solution $\alpha = 0$, $\beta = 0$, q.e.d. The conjecture is completely verified and the maximal multiplicity is 10.

## 7. Description of a center set for $p$, $q$ of small degrees.

Consider again the polynomial Abel equation (1.2):

$$y' = p(x)y^2 + q(x)y^3, \qquad y(0) = y_0$$

with $p(x), q(x)$ – polynomials in $x$ of the degrees $d_1, d_2$ respectively. We will write

$$p(x) = \lambda_{d_1} x^{d_1} + \cdots + \lambda_0,$$

$$q(x) = \mu_{d_2} x^{d_2} + \cdots + \mu_0,$$

$$(\lambda_{d_1}, \ldots, \lambda_0, \mu_{d_1}, \ldots, \mu_0) = (\lambda, \mu) \in \mathbb{C}^{d_1+d_2+2}.$$

Remind that $v_k(x)$ (see Introduction for details) are polynomials in $x$ with the coefficients polynomially depending on the parameters $(\lambda, \mu) \in \mathbb{C}^{d_1+d_2+2}$. Let the **center set** $C \subset \mathbb{C}^{d_1+d_2+2}$ consist of those $(\lambda, \mu)$ for which $y(0) \equiv y(1)$ for all the solutions $y(x)$ of (1.2). (This definition is not completely accurate, since the value $y(1)$ may depend on the continuation path from 0 to 1 in the $x$-plane. See section 2 for detailed discussion.)

Clearly, $C$ is defined by an infinite number of polynomial equations in $(\lambda, \mu)$ : $v_2(1) = 0, \ldots, v_k(1) = 0, \ldots$ In other words, $C$ is the set of zeroes $Y(I)$ of the ideal $I = \{v_1(1), \ldots, v_k(1), \ldots\}$ in the ring of polynomials $\mathbb{C}[\lambda, \mu]$. (In [BFY1] $I$ is called the Bautin ideal of the equation (1.2).) Notice that in this section, in contrast to the general approach introduced in this paper, we consider $I$ as the ideal in $\mathbb{C}[\lambda, \mu]$ and not in $\mathbb{C}[x, \lambda, \mu]$ .

The table of multiplicities, given in the theorem 6.1, gives the number of equation $v_k(1) = 0$, necessary to define $C$ (i.e. the stabilization moment for the set of zeroes of the ideals $I_k(x)$). Since both $v_k(1)$ and $\psi_k(1)$ are polynomials of degree $k-1$ in $(\lambda, \mu)$, the straightforward description of $C$ contains polynomials of a rather high degree, for example up to degree 10 of 9 variables for the case $(\deg P, \deg Q) = (3, 6)$.

As it was said before, the composition conjecture, in contrast, gives us very explicit and transparent equations, describing this center set $C$. Especially explicit are equations in a parametric form (see below).

**7.1.** The central set for the equation (1.2) with deg $p$ = deg $q$ = 2 has been described in [BFY1]. We remind this result here. Let

$$\begin{aligned} p(x) &= \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

**Theorem 7.1.** ([BFY1], Theorem V.1) *The center set $C \subseteq \mathbb{C}^6$ of the equation (1.2) is given by*

$$\begin{aligned} 2\lambda_2 + 3\lambda_1 + 6\lambda_0 &= 0 \\ 2\mu_2 + 3\mu_1 + 6\mu_0 &= 0 \\ \lambda_2\mu_1 - \lambda_1\mu_2 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^6$ is determined by vanishing of the first 3 Taylor coefficients $v_2(1) = 0, \ldots, v_4(1) = 0$*

Of course, this result, which was obtained from completely different considerations than in this article, confirms the composition conjecture: since $P$ and $Q$ are of a prime degree 3, their greater common divisor in a composition algebra can be either $x$ or a polynomial of degree 3. This corresponds to a proportionality of $P$ and $Q$ (or of $p$ and $q$), which gives us exactly the last equation, and the first two are obtained from $P(1) = Q(1) = 0$

**7.2.** Now let

$$\begin{aligned} p(x) = &\quad \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) = &\quad \mu_1 x + \mu_0 \end{aligned}$$

**Theorem 7.2.** *The center set $C \subseteq \mathbb{C}^6$ of the equation (1.2) is given by*

$$\begin{aligned} 2\lambda_2 + 3\lambda_3 &= 0 \\ 2\lambda_1 - \lambda_3 + 4\lambda_0 &= 0 \\ \mu_1 + 2\mu_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^6$ is determined by vanishing of the first 3 Taylor coefficients $v_2(1) = 0, \ldots, v_4(1) = 0$.*

**Proof:**

By the composition conjecture, which holds for this case, $p$ and $q$ belong to the center set if and only if $P = \tilde{P}(W)$, $Q = \mu W$ , where $W = x(x-1)$. So, we may assume $Q = \mu x(x-1)$, $P = \alpha W^2 + \beta W$. Thus we get

$$Q = \mu x(x-1) = \frac{\mu_1}{2} x^2 + \mu_0 x$$

$$P = \alpha \left(x(x-1)\right)^2 + \beta x(x-1) = \frac{\lambda_3}{4} x^4 + \frac{\lambda_2}{3} x^3 + \frac{\lambda_1}{2} x^2 + \lambda_0 x$$

Comparing coefficients of $x^k$ in both sides of equalities, we get

$$\begin{cases} \lambda_3 = 4\alpha & \lambda_2 = -6\alpha \\ \lambda_1 = 2\alpha + 2\beta & \lambda_0 = -\beta \\ \mu_1 = 2\mu & \mu_0 = -\mu \end{cases},$$

which is equivalent to the system in the statement of the theorem.

**7.3.** If

$$\begin{aligned} p(x) = &\quad \lambda_1 x + \lambda_0 \\ q(x) = &\quad \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

then similarly to the previous theorem one can prove the following

**Theorem 7.3.** *The center set $C \subseteq \mathbb{C}^6$ of the equation (1.2) is given by*

$$\begin{aligned} 2\mu_2 + 3\mu_3 &= 0 \\ 2\mu_1 - \mu_3 + 4\mu_0 &= 0 \\ \lambda_1 + 2\lambda_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^6$ is determined by vanishing of the first 3 Taylor coefficients $v_2(1) = 0, \ldots, v_4(1) = 0$.*

**7.4.** Now let

$$\begin{aligned} p(x) &= \lambda_5 x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_1 x + \mu_0 \end{aligned}$$

**Theorem 7.4.** *The center set $C \subseteq \mathbb{C}^8$ of the equation (1.2) is given by*

$$\begin{aligned} 5\lambda_5 + 2\lambda_4 &= 0 \\ 10\lambda_5 + 12\lambda_4 + 15\lambda_3 + 20\lambda_2 + 30\lambda_1 + 60\lambda_0 &= 0 \\ \lambda_3 + 4\lambda_2 + 10\lambda_1 + 20\lambda_0 &= 0 \\ \mu_1 + 2\mu_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^8$ is determined by vanishing of the first 8 Taylor coefficients $v_2(1) = 0, \ldots, v_9(1) = 0$.*

**Proof:**

By the composition conjecture, which holds for this case, $p$ and $q$ belong to the center set if and only if $P = \tilde{P}(W)$, $Q = \mu W$ , where $W = x(x-1)$. So, we may assume $Q = \mu x(x-1)$, $P = \alpha W^3 + \beta W^2 + \gamma W$. Thus we get

$$\mu x(x-1) = \frac{\mu_1}{2} x^2 + \mu_0 x$$

$$\alpha \left(x(x-1)\right)^2 + \beta \left(x(x-1)\right)^2 + \gamma x(x-1) = \frac{\lambda_5}{6} x^6 + \frac{\lambda_4}{5} x^5 + \frac{\lambda_3}{4} x^4 + \frac{\lambda_2}{3} x^3 + \frac{\lambda_1}{2} x^2 + \lambda_0 x$$

Hence

$$\begin{cases} \lambda_5 = 6\alpha & \lambda_4 = -15\alpha \\ \lambda_3 = 12\alpha + 4\beta & \lambda_2 = -3\alpha - 6\beta \\ \lambda_1 = 2\beta + 2\gamma & \lambda_0 = -\gamma \\ \mu_1 = 2\mu & \mu_0 = -\mu \end{cases},$$

which is equivalent to the system in the statement of the theorem.

**7.5.** If

$$\begin{aligned} p(x) &= \lambda_1 x + \lambda_0 \\ q(x) &= \mu_5 x^5 + \mu_4 x^4 + \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

then similarly to the previous theorem one can prove the following

**Theorem 7.5.** *The center set $C \subseteq \mathbb{C}^8$ of the equation (1.2) is given by*

$$\begin{aligned} 5\mu_5 + 2\mu_4 &= 0 \\ 10\mu_5 + 12\mu_4 + 15\mu_3 + 20\mu_2 + 30\mu_1 + 60\mu_0 &= 0 \\ \mu_3 + 4\mu_2 + 10\mu_1 + 20\mu_0 &= 0 \\ \lambda_1 + 2\lambda_0 &= 0 \end{aligned}$$

*The set $C$ in $\mathbb{C}^8$ is determined by vanishing of the first 4 Taylor coefficients $v_2(1) = 0, \dots, v_5(1) = 0$.*

**Remark:** An interesting fact that center sets $C$ for the "similar" cases $\deg p = 5, \deg q = 1$ and $\deg p = 1, \deg q = 5$ have different number of generators $v_k(1) = 0$ can be explained by a different role of $p$ and $q$ in ideals $I$. See section 3 for the first ideals $I_k$, and [BFY2] for an attempt to analyze this problem.

**7.6.** Now let

$$p(x) = \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0,$$

$$q(x) = \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0.$$

**Theorem 7.6.** ([BFY2], Theorem 9.2.) *The central set $C \subseteq \mathbb{C}^8$ of (1.2) consists of two components $C^{(1)}$ and $C^{(2)}$, each of dimension 4.*

*$C^{(1)}$ is given by*

$$\begin{cases} 3\lambda_3 + 4\lambda_2 + 6\lambda_1 + 12\lambda_0 = 0 \\ 3\mu_3 + 4\mu_2 + 6\mu_1 + 12\mu_0 = 0 \end{cases} \tag{7.6.1}$$

*and*

$$\begin{cases} \lambda_3\mu_2 - \mu_3\lambda_2 = 0 \\ \lambda_3\mu_1 - \mu_3\lambda_1 = 0 \\ \lambda_2\mu_1 - \mu_2\lambda_1 = 0 \end{cases} \tag{7.6.2}$$

*and $C^{(2)}$ is given by* (7.6.1) *and*

$$\begin{cases} 3\lambda_3 + 2\lambda_2 = 0 \\ 3\mu_3 + 2\mu_2 = 0 \end{cases} \tag{7.6.3}$$

*The set $C$ in $\mathbb{C}^8$ is determined by the vanishing of the first 8 Taylor coefficients $v_2(1) = 0, \dots, v_9(1) = 0$.*

This theorem was proved in [BFY2] using the fact that the composition conjecture is true for this case. The component (7.6.1 & 7.6.2) corresponds to the proportionality of $P$ and $Q$, and the component (7.6.1 & 7.6.3) corresponds to the composition with $W = x(x-1)$.

**7.7.** Let

$$\begin{aligned} p(x) &= \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ q(x) &= \mu_5 x^5 + \mu_4 x^4 + \mu_3 x^3 + \mu_2 x^2 + \mu_1 x + \mu_0 \end{aligned}$$

then similarly to the previous theorems one can prove the following

**Theorem 7.7.** *The center set $C \subseteq \mathbb{C}^9$ of the equation (1.2) is given in a parametric form by*

$$\begin{cases} \lambda_2 = 3\lambda & \lambda_1 = -2\lambda(a+1) \\ \lambda_0 = a\lambda & \mu_5 = 6\alpha \\ \mu_4 = -10\alpha(a+1) & \mu_3 = 4\alpha(a+1)^2 + 8a\alpha \\ \mu_2 = -6\alpha a(a+1) + 3\beta & \mu_1 = 2\alpha a^2 - 2(a+1)\beta \\ \mu_0 = a\beta \end{cases} \tag{7.7.1}$$

*or by*

$$\begin{cases} \mu_4 = -\dfrac{5\mu_5}{3}\left(\dfrac{3\lambda_0}{\lambda_2} + 1\right) \\[2ex] \mu_3 = \dfrac{2\mu_5}{3}\left(\dfrac{3\lambda_0}{\lambda_2} + 1\right)^2 + 4\dfrac{\lambda_0}{\lambda_2}\mu_5 \\[2ex] \mu_2 = -\dfrac{3\mu_5\lambda_0}{\lambda_2}\left(\dfrac{3\lambda_0}{\lambda_2} + 1\right) + \dfrac{\mu_0\lambda_2}{\lambda_0} \\[2ex] \mu_1 = \dfrac{3\mu_5\lambda_0^2}{\lambda_2^2} - 2\left(\dfrac{3\lambda_0}{\lambda_2} + 1\right)\dfrac{\mu_0\lambda_2}{3\lambda_0} \\[2ex] 3\lambda_1 = -2\lambda_2 - 6\lambda_0 \end{cases} \tag{7.7.2}$$

*The set $C$ in $\mathbb{C}^9$ is determined by the vanishing of the first 9 Taylor coefficients $v_2(1) = 0, \ldots, v_{10}(1) = 0$.*

**Proof:**
We can represent $P = \lambda W$, $Q = \alpha W^2 + \beta W$, where $W = x(x-1)(x-a)$. Thus

$$\lambda\left(x^3 - (a+1)x^2 + ax\right) = \frac{\lambda_2}{3}x^3 + \frac{\lambda_1}{2}x^2 + \lambda_0 x,$$

$$\alpha\left(x^6 + (a+1)^2x^4 + a^2x^2 - 2(a+1)x^5 + 2ax^4 - 2a(a+1)x^3\right) +$$

$$\beta\left(x^3 - x^2(a+1) + ax\right) = \frac{\mu_5}{6}x^6 + \ldots + \mu_0 x$$

Comparing coefficients by $x^k$ in both sides of equalities, we obtain (7.7.1). After some transformations we obtain (7.7.2). (Notice, that $\lambda_2 \neq 0$ as leading coefficient.)

**Remark:** Essential nonlinearity in (7.7.2) appears because of a "free period" $a$. Notice, that for fixed $a$ the parametric form (7.7.1) is linear with respect to $\alpha, \beta, \lambda$. One can notice, that nonlinearity appears in those and only those cases, when there are "moving periods", different from the endpoint 1 (see 7.1, 7.6, 7.7).

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